Sequentially Cohen-Macaulay Rees modules

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October 29, 2014

Introduction

[CGT]

N. T. Cuong, S. Goto and H. L. Truong, *The equality $I^2 = \mathfrak{q}I$ in sequentially Cohen-Macaulay rings*, J. Algebra, **(379)** (2013), 50-79.

In [CGT],

- Characterized the sequentially Cohen-Macaulayness of $\mathcal{R}(I)$ where I is an $\mathfrak{m}$-primary ideal which contains a good parameter ideal as a reduction. ([Theorem 5.3]).

Question 1.1

When is the Rees module $\mathcal{R}(\mathcal{M})$ sequentially Cohen-Macaulay?

Contents

Filtration

Let R be a commutative ring.

Definition 2.1

$\mathcal{F} = \{F_n\}_{n\in\mathbb{Z}}$ is a filtration of ideals of R
$\overset{def}{\Longleftrightarrow}$

1. F_n is an ideal of R,
2. $F_n \supseteq F_{n+1}$ for $\forall n \in \mathbb{Z}$,
3. $F_m F_n \subseteq F_{m+n}$ for $\forall m, n \in \mathbb{Z}$ and
4. $F_0 = R$.

Then we put

$$\mathcal{R} = \mathcal{R}(\mathcal{F}) = \sum_{n\geq 0} F_n t^n \subseteq R[t], \quad \mathcal{R}' = \mathcal{R}'(\mathcal{F}) = \sum_{n\in\mathbb{Z}} F_n t^n \subseteq R[t, t^{-1}].$$

Let M be an R-module.

Definition 2.2

$\mathcal{M} = \{M_n\}_{n \in \mathbb{Z}}$ is an $\mathcal{F}$-filtration of R-submodules of M

$\overset{def}{\Longleftrightarrow}$

1. M_n is an R-submodule of M,
2. $M_n \supseteq M_{n+1}$ for $\forall n \in \mathbb{Z}$,
3. $F_m M_n \subseteq M_{m+n}$ for $\forall m, n \in \mathbb{Z}$ and
4. $M_0 = M$.

We set

$$\mathcal{R}(\mathcal{M}) = \sum_{n \geq 0} t^n \otimes M_n \subseteq R[t] \otimes_R M,$$

$$\mathcal{R}'(\mathcal{M}) = \sum_{n \in \mathbb{Z}} t^n \otimes M_n \subseteq R[t, t^{-1}] \otimes_R M.$$

Here

$$t^n \otimes M_n = \{t^n \otimes x \mid x \in M_n\} \subseteq R[t, t^{-1}] \otimes_R M$$

for $\forall n \in \mathbb{Z}$.

If $F_1 \neq R$, then we put

$$\mathcal{G} = \mathcal{G}(\mathcal{F}) = \mathcal{R}'/u\mathcal{R}', \quad \mathcal{G}(\mathcal{M}) = \mathcal{R}'(\mathcal{M})/u\mathcal{R}'(\mathcal{M})$$

where $u = t^{-1}$.

For the rest of this section, we assume $F_1 \neq R$.

Lemma 2.3

Suppose R is Noetherian and M is finitely generated. Then TFAE.

(1) $\mathcal{R}(\mathcal{M})$ *is a finitely generated graded* $\mathcal{R}$*-module.*

(2) $\mathcal{R}'(\mathcal{M})$ *is a finitely generated graded* $\mathcal{R}'$*-module.*

(3) $\exists n_1, n_2, \ldots, n_\ell \geq 0$ $(\ell > 0)$ *s.t.* $M_n = \sum_{i=1}^{\ell} F_{n-n_i} M_{n_i}$ *for* $\forall n \geq \max\{n_1, n_2, \ldots, n_\ell\}$.

- The composite map

$$\psi : \mathcal{R}(\mathcal{M}) \xrightarrow{i} \mathcal{R}'(\mathcal{M}) \xrightarrow{\varepsilon} \mathcal{G}(\mathcal{M})$$

is surjective and

- $$\operatorname{Ker} \psi = u\mathcal{R}'(\mathcal{M}) \cap \mathcal{R}(\mathcal{M}) = u[\mathcal{R}(\mathcal{M})]_+,$$

where $[\mathcal{R}(\mathcal{M})]_+ = \sum_{n>0} t^n \otimes M_n$.

Assumption 2.4

- $\mathcal{R}(\mathcal{F})$ a Noetherian ring
- $\mathcal{R}(\mathcal{M})$ a finitely generated $\mathcal{R}$-module

Then R is Noetherian and M is finitely generated.

Proposition 2.5

The following assertions hold true.

(1) *Let* $P \in \mathrm{Ass}_{\mathcal{R}} \mathcal{R}(\mathcal{M})$. *Then* $\mathfrak{p} \in \mathrm{Ass}_R M$, $P = \mathfrak{p}R[t] \cap \mathcal{R}$ *and*

$$\dim \mathcal{R}/P = \begin{cases} \dim R/\mathfrak{p} + 1 & \textit{if } \dim R/\mathfrak{p} < \infty, F_1 \not\subseteq \mathfrak{p}, \\ \dim R/\mathfrak{p} & \textit{otherwise}, \end{cases}$$

where $\mathfrak{p} = P \cap R$.

(2) *Suppose* $M \neq (0)$, $d = \dim_R M < \infty$ *and* $\exists \mathfrak{p} \in \mathrm{Ass}_R M$ *s.t.* $\dim R/\mathfrak{p} = d$, $F_1 \not\subseteq \mathfrak{p}$. *Then* $\dim_{\mathcal{R}} \mathcal{R}(\mathcal{M}) = d + 1$.

Proof.

(1) Let $P \in \operatorname{Ass}_{\mathcal{R}} \mathcal{R}(\mathcal{M})$. Then $P \in \operatorname{Ass}_{\mathcal{R}} R[t] \otimes_R M$, so that $P = Q \cap \mathcal{R}$ for some

$$Q \in \operatorname{Ass}_{R[t]} R[t] \otimes_R M = \bigcup_{\mathfrak{p} \in \operatorname{Ass}_R M} \operatorname{Ass}_{R[t]} R[t]/\mathfrak{p}R[t].$$

Thus $\mathfrak{p} = Q \cap R$ and $Q = \mathfrak{p}R[t]$ for $\exists \mathfrak{p} \in \operatorname{Ass}_R M$. Therefore

$$P = \mathfrak{p}R[t] \cap \mathcal{R}, \ \mathfrak{p} = P \cap R.$$

Let $\overline{R} = R/\mathfrak{p}$. Then $\overline{\mathcal{F}} = \{F_n\overline{R}\}_{n \in \mathbb{Z}}$ is a filtration of ideals of $\overline{R}$ and

$$\mathcal{R}/P \cong \mathcal{R}(\overline{\mathcal{F}})$$

as graded R-algebras. □

Corollary 2.6

Suppose R is local, $M \neq (0)$. Then

$$\dim_{\mathcal{R}} \mathcal{R}(\mathcal{M}) = \begin{cases} d+1 & \text{if } \exists \mathfrak{p} \in \operatorname{Ass}_R M \text{ s.t. } \dim R/\mathfrak{p} = d, F_1 \not\subseteq \mathfrak{p}, \\ d & \text{otherwise}, \end{cases}$$

where $d = \dim_R M$.

Proposition 2.7

The following assertions hold true.

(1) *Let $P \in \operatorname{Ass}_{\mathcal{R}'} \mathcal{R}'(\mathcal{M})$. Then $\mathfrak{p} \in \operatorname{Ass}_R M$, $P = \mathfrak{p}R[t, t^{-1}] \cap \mathcal{R}'$ and $\dim \mathcal{R}'/P = \dim R/\mathfrak{p} + 1$, where $\mathfrak{p} = P \cap R$.*

(2) *Suppose $M \neq (0)$. Then $\dim_{\mathcal{R}'} \mathcal{R}'(\mathcal{M}) = \dim_R M + 1$.*

Lemma 2.8

Suppose that R *is a local ring,* $M \neq (0)$. *Then* $\mathcal{G}(\mathcal{M}) \neq (0)$ *and* $\dim_{\mathcal{G}} \mathcal{G}(\mathcal{M}) = \dim_R M$.

Proof.

Let $\mathfrak{N}$ be a unique graded maximal ideal of an H-local ring $\mathcal{R}'$. Then $\mathcal{R}'(\mathcal{M})_{\mathfrak{N}} \neq (0)$ and $u \in \mathfrak{N}$. Therefore $\mathcal{G}(\mathcal{M})_{\mathfrak{N}} \neq (0)$, so that $\mathcal{G}(\mathcal{M}) \neq (0)$. Hence $\dim_{\mathcal{G}} \mathcal{G}(\mathcal{M}) = \dim_R M$. □

Survey on sequentially C-M modules

Let R be a Noetherian ring and $M \neq (0)$ a finitely generated R-module with $d = \dim_R M < \infty$. We put

$$\operatorname{Assh}_R M = \{\mathfrak{p} \in \operatorname{Supp}_R M \mid \dim R/\mathfrak{p} = d\}.$$

Then $\forall n \in \mathbb{Z}$, $\exists M_n$ the largest R-submodule of M with $\dim_R M_n \leq n$. Let

$$\begin{aligned} \mathcal{S}(M) &= \{\dim_R N \mid N \text{ is an } R\text{-submodule of } M, N \neq (0)\} \\ &= \{\dim R/\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Ass}_R M\} \\ &= \{d_1 < d_2 < \cdots < d_\ell = d\} \end{aligned}$$

where $\ell = \sharp\mathcal{S}(M)$.

Let $D_i = M_{d_i}$ for $1 \leq \forall i \leq \ell$. We then have a filtration

$$\mathcal{D} : D_0 := (0) \subsetneq D_1 \subsetneq D_2 \subsetneq \ldots \subsetneq D_\ell = M$$

which we call *the dimension filtration of M*. Put $C_i = D_i / D_{i-1}$ for $1 \leq \forall i \leq \ell$.

Definition 3.1 ([Sch, St])

(1) M is *a sequentially Cohen-Macaulay R-module*
$\overset{def}{\Longleftrightarrow}$ C_i is a C-M R-module for $1 \leq \forall i \leq \ell$.

(2) R is *a sequentially Cohen-Macaulay ring*
$\overset{def}{\Longleftrightarrow}$ $\dim R < \infty$ and R is a sequentially C-M module over itself.

Let

$$(0) = \bigcap_{\mathfrak{p} \in \operatorname{Ass}_R M} M(\mathfrak{p})$$

be a primary decomposition of (0) in M, where $\operatorname{Ass}_R M/M(\mathfrak{p}) = \{\mathfrak{p}\}$ for $\forall \mathfrak{p} \in \operatorname{Ass}_R M$.

Fact 3.2 ([Sch])

The following assertions hold true.

(1) $D_i = \bigcap_{\dim R/\mathfrak{p} \geq d_{i+1}} M(\mathfrak{p})$ *for* $0 \leq \forall i < \ell$.

(2) $\operatorname{Ass}_R C_i = \{\mathfrak{p} \in \operatorname{Ass}_R M \mid \dim R/\mathfrak{p} = d_i\}$ *and* $\operatorname{Ass}_R D_i = \{\mathfrak{p} \in \operatorname{Ass}_R M \mid \dim R/\mathfrak{p} \leq d_i\}$ *for* $1 \leq \forall i \leq \ell$.

Theorem 3.3 ([GHS])

Let $\mathcal{M} = \{M_i\}_{0 \le i \le t}$ $(t > 0)$ *be a family of* R*-submodules of* M *s.t.*

(1) $M_0 = (0) \subsetneq M_1 \subsetneq M_2 \subsetneq \ldots \subsetneq M_t = M$ *and*

(2) $\dim_R M_{i-1} < \dim_R M_i$ *for* $1 \le \forall i \le t$.

Assume that $\operatorname{Ass}_R M_i/M_{i-1} = \operatorname{Assh}_R M_i/M_{i-1}$ *for* $1 \le \forall i \le t$. *Then* $t = \ell$ *and* $M_i = D_i$ *for* $0 \le \forall i \le \ell$.

Proposition 3.4 (NZD characterization)

Let $(R, \mathfrak{m})$ be a Noetherian local ring, $M \neq (0)$ a finitely generated R-module. Let $x \in \mathfrak{m}$ be a NZD on M. Then TFAE.

(1) *M is a sequentially C-M R-module.*

(2) *M/xM is a sequentially C-M $R/(x)$-module and $\{D_i/xD_i\}_{0 \leq i \leq \ell}$ is the dimension filtration of M/xM.*

Proof.

Since $x \in \mathfrak{m}$ is a NZD on C_i and on D_i for $1 \leq \forall i \leq \ell$, so that we get a filtration

$$D_0/xD_0 = (0) \subsetneq D_1/xD_1 \subsetneq \cdots \subsetneq D_\ell/xD_\ell = M/xM.$$

□

Remark 3.5

The implication $(2) \Rightarrow (1)$ is not true without the condition that $\{D_i/xD_i\}_{0\leq i\leq \ell}$ is the dimension filtration of M/xM.

For example, let R be a 2-dimensional Noetherian local domain of depth 1 (Nagata's bad example). Then $R/(x)$ is sequentially C-M for $x \neq 0$, but R is not sequentially C-M.

This example shows that [Sch, Theorem 4.7] is not true in general.

Main results

Notation 4.1

- $(R, \mathfrak{m})$ *a Noetherian local ring*
- $M \neq (0)$ *a finitely generated* R*-module with* $d = \dim_R M$
- $\mathcal{F} = \{F_n\}_{n \in \mathbb{Z}}$ *a filtration of ideals of* R *s.t.* $F_1 \neq R$
- $\mathcal{M} = \{M_n\}_{n \in \mathbb{Z}}$ *an* $\mathcal{F}$*-filtration of* R*-submodules of* M
- $\mathfrak{a} = \mathcal{R}(\mathcal{F})_+ = \sum_{n>0} F_n t^n$
- $\mathfrak{M}$ *a unique graded maximal ideal of* $\mathcal{R}$
- $\mathcal{R} = \mathcal{R}(\mathcal{F})$ *a Noetherian ring*
- $\mathcal{R}(\mathcal{M})$ *a finitely generated* $\mathcal{R}$*-module*

Let $1 \le i \le \ell$. We set

$$\mathcal{D}_i = \{M_n \cap D_i\}_{n \in \mathbb{Z}}, \quad \mathcal{C}_i = \{[(M_n \cap D_i) + D_{i-1}]/D_{i-1}\}_{n \in \mathbb{Z}}.$$

Then $\mathcal{D}_i$ (resp. $\mathcal{C}_i$) is an $\mathcal{F}$-filtration of R-submodules of D_i (resp. C_i). Look at the exact sequence

$$0 \to [\mathcal{D}_{i-1}]_n \to [\mathcal{D}_i]_n \to [\mathcal{C}_i]_n \to 0$$

of R-modules for $\forall n \in \mathbb{Z}$. We then have

$$0 \to \mathcal{R}(\mathcal{D}_{i-1}) \to \mathcal{R}(\mathcal{D}_i) \to \mathcal{R}(\mathcal{C}_i) \to 0$$

$$0 \to \mathcal{R}'(\mathcal{D}_{i-1}) \to \mathcal{R}'(\mathcal{D}_i) \to \mathcal{R}'(\mathcal{C}_i) \to 0 \quad \text{and}$$

$$0 \to \mathcal{G}(\mathcal{D}_{i-1}) \to \mathcal{G}(\mathcal{D}_i) \to \mathcal{G}(\mathcal{C}_i) \to 0.$$

Theorem 4.2

TFAE.

(1) $\mathcal{R}'(\mathcal{M})$ *is a sequentially C-M* $\mathcal{R}'$*-module.*

(2) $\mathcal{G}(\mathcal{M})$ *is a sequentially C-M* $\mathcal{G}$*-module and* $\{\mathcal{G}(\mathcal{D}_i)\}_{0\le i\le \ell}$ *is the dimension filtration of* $\mathcal{G}(\mathcal{M})$*.*

When this is the case, M *is a sequentially C-M* R*-module.*

Theorem 4.3

Suppose that M *is a sequentially C-M* R*-module and* $F_1 \not\subseteq \mathfrak{p}$ *for* $\forall \mathfrak{p} \in \operatorname{Ass}_R M$*. Then TFAE.*

(1) $\mathcal{R}(\mathcal{M})$ *is a sequentially C-M* $\mathcal{R}$*-module.*

(2) $\mathcal{G}(\mathcal{M})$ *is a sequentially C-M* $\mathcal{G}$*-module,* $\{\mathcal{G}(\mathcal{D}_i)\}_{0\le i\le \ell}$ *is the dimension filtration of* $\mathcal{G}(\mathcal{M})$ *and* $\mathrm{a}(\mathcal{G}(\mathcal{C}_i)) < 0$ *for* $1 \le \forall i \le \ell$*.*

When this is the case, $\mathcal{R}'(\mathcal{M})$ *is a sequentially C-M* $\mathcal{R}'$*-module.*

Lemma 4.4 (cf. [CGT])

$\{\mathcal{R}'(\mathcal{D}_i)\}_{0\le i\le \ell}$ is the dimension filtration of $\mathcal{R}'(\mathcal{M})$. If $F_1 \nsubseteq \mathfrak{p}$ for $\forall \mathfrak{p} \in \operatorname{Ass}_R M$, then $\{\mathcal{R}(\mathcal{D}_i)\}_{0\le i\le \ell}$ is the dimension filtration of $\mathcal{R}(\mathcal{M})$.

Proof.

Look at the filtration

$$\mathcal{R}'(\mathcal{D}_0) = (0) \subsetneq \mathcal{R}'(\mathcal{D}_1) \subsetneq \mathcal{R}'(\mathcal{D}_2) \subsetneq \cdots \subsetneq \mathcal{R}'(\mathcal{D}_\ell) = \mathcal{R}'(\mathcal{M}).$$

Then $\dim_{\mathcal{R}'} \mathcal{R}'(\mathcal{D}_i) = d_i + 1$. Let $P \in \operatorname{Ass}_{\mathcal{R}'} \mathcal{R}'(\mathcal{C}_i)$. Then we have

$$\dim \mathcal{R}'/P = d_i + 1 = \dim_{\mathcal{R}'} \mathcal{R}'(\mathcal{C}_i)$$

by Proposition 2.7. Therefore $\{\mathcal{R}'(\mathcal{D}_i)\}_{0\le i\le \ell}$ is the dimension filtration of $\mathcal{R}'(\mathcal{M})$. □

Proof of Theorem 4.2

Look at the exact sequence

$$0 \to \mathcal{R}'(\mathcal{C}_i) \to R[t, t^{-1}] \otimes_R C_i \to X \to 0$$

of graded $\mathcal{R}'$-modules for $1 \leq i \leq \ell$.

Since $\mathcal{R}'(\mathcal{C}_i)$ is C-M and $X_u = (0)$, we have $R[t, t^{-1}] \otimes_R C_i$ is C-M.

Therefore M is sequentially C-M, because C_i is C-M.

□

Towards a proof of Theorem 4.3

Fact 4.5 ([F])

Let I *be an ideal of* R *and* $t \in \mathbb{Z}$. *Consider the following two conditions.*

(1) $\exists \ell > 0$ *s.t.* $I^{\ell} {\cdot} \mathrm{H}^i_{\mathfrak{m}}(M) = (0)$ *for* $\forall i \neq t$.

(2) $M_{\mathfrak{p}}$ *is a C-M* $R_{\mathfrak{p}}$*-module and* $t = \dim_{R_{\mathfrak{p}}} M_{\mathfrak{p}} + \dim R/\mathfrak{p}$ *for* $\forall \mathfrak{p} \in \operatorname{Supp}_R M$ *but* $\mathfrak{p} \not\supseteq I$.

Then the implication $(1) \Rightarrow (2)$ *holds true. The converse holds, if* R *is a homomorphic image of a Gorenstein local ring.*

Lemma 4.6 (Key lemma)

Suppose that $\mathrm{H}^i_{\mathfrak{M}}(\mathcal{G}(\mathcal{M}))$ *is finitely graded for* $\forall i \neq d$. *Then* $\mathrm{H}^i_{\mathfrak{M}}(\mathcal{R}(\mathcal{M}))$ *is finitely graded for* $\forall i \neq d+1$.

Proof of Lemma 4.6

It is enough to show that

$$\exists \ell > 0 \ \text{ s.t. } \ \mathfrak{a}^\ell \cdot \mathrm{H}^i_{\mathfrak{M}}(\mathcal{R}(M)) = (0) \ \text{ for } \ i \neq d+1.$$

To see this, let $P \in \mathrm{Supp}_{\mathcal{R}} \mathcal{R}(M)$ s.t. $P \not\supseteq \mathfrak{a}$ and $P \subseteq \mathfrak{M}$.

Put $L = u\mathfrak{a} = u\mathcal{R}' \cap \mathcal{R}$.

Proof of Lemma 4.6

Fact 4.7

$\sqrt{P^* + L} \not\supseteq \mathfrak{a}$.

Therefore $\exists Q \in \operatorname{Min}_{\mathcal{R}} \mathcal{R}/[P^* + L]$ s.t $\mathfrak{a} \not\subseteq Q \subseteq \mathfrak{M}$. Then we can show that $\mathcal{G}(\mathcal{M})_Q$ is C-M,

$$d = \dim_{\mathcal{R}_Q} \mathcal{G}(\mathcal{M})_Q + \dim \mathcal{R}_{\mathfrak{M}}/Q\mathcal{R}_{\mathfrak{M}}.$$

Hence $\mathcal{R}(\mathcal{M})_Q$ is C-M and

$$d + 1 = \dim_{\mathcal{R}_Q} \mathcal{R}(\mathcal{M})_Q + \dim \mathcal{R}_{\mathfrak{M}}/Q\mathcal{R}_{\mathfrak{M}}.$$

Proof of Lemma 4.6

Since $P^* \subseteq Q$, $\mathcal{R}(M)_{P^*}$ is C-M, so is $\mathcal{R}(M)_P$. We also have

$$d+1 = \dim_{\mathcal{R}_P} \mathcal{R}(M)_P + \dim \mathcal{R}_{\mathfrak{M}}/P\mathcal{R}_{\mathfrak{M}}.$$

Thanks to Fact 4.5, $\exists \ell > 0$ s.t.

$$\mathfrak{a}^{\ell} \cdot \mathrm{H}^i_{\mathfrak{M}}(\mathcal{R}(\mathcal{M})) = (0) \quad \text{for} \quad i \neq d+1$$

which shows $\mathrm{H}^i_{\mathfrak{M}}(\mathcal{R}(\mathcal{M}))$ is finitely graded. □

We set

$$\mathrm{a}(N) = \max\{n \in \mathbb{Z} \mid [\mathrm{H}^t_{\mathfrak{M}}(N)]_n \neq (0)\}$$

for a finitely generated graded $\mathcal{R}$-module N of dimension t, and call it *the a-invariant of* N ([GW]).

Theorem 4.8

TFAE.

(1) $\mathcal{R}(\mathcal{M})$ *is a C-M* $\mathcal{R}$*-module and* $\dim_{\mathcal{R}} \mathcal{R}(\mathcal{M}) = d + 1$.

(2) $\mathrm{H}^i_{\mathfrak{M}}(\mathcal{G}(\mathcal{M})) = [\mathrm{H}^i_{\mathfrak{M}}(\mathcal{G}(\mathcal{M}))]_{-1}$ *for* $\forall i < d$ *and* $\mathrm{a}(\mathcal{G}(\mathcal{M})) < 0$.

When this is the case, $[\mathrm{H}^i_{\mathfrak{M}}(\mathcal{G}(\mathcal{M}))]_{-1} \cong \mathrm{H}^i_{\mathfrak{m}}(M)$ *for* $\forall i < d$.

Corollary 4.9

Suppose that M is a C-M R-module. Then TFAE.

(1) $\mathcal{R}(\mathcal{M})$ *is a C-M* $\mathcal{R}$*-module and* $\dim_{\mathcal{R}} \mathcal{R}(\mathcal{M}) = d + 1$.

(2) $\mathcal{G}(\mathcal{M})$ *is a C-M* $\mathcal{G}$*-module and* $\mathrm{a}(\mathcal{G}(\mathcal{M})) < 0$.

Theorem 4.3

Suppose that M is a sequentially C-M R-module and $F_1 \not\subseteq \mathfrak{p}$ for $\forall \mathfrak{p} \in \mathrm{Ass}_R M$. Then TFAE.

(1) $\mathcal{R}(\mathcal{M})$ *is a sequentially C-M* $\mathcal{R}$*-module.*

(2) $\mathcal{G}(\mathcal{M})$ *is a sequentially C-M* $\mathcal{G}$*-module,* $\{\mathcal{G}(\mathcal{D}_i)\}_{0 \le i \le \ell}$ *is the dimension filtration of* $\mathcal{G}(\mathcal{M})$ *and* $\mathrm{a}(\mathcal{G}(\mathcal{C}_i)) < 0$ *for* $1 \le \forall i \le \ell$.

When this is the case, $\mathcal{R}'(\mathcal{M})$ *is a sequentially C-M* $\mathcal{R}'$*-module.*

Proof of Theorem 4.3

$\mathcal{R}(\mathcal{M})$ is a sequentially C-M $\mathcal{R}$-module

$\Longleftrightarrow \mathcal{R}(\mathcal{C}_i)$ is a C-M $\mathcal{R}$-module for $1 \leq \forall i \leq \ell$

$\Longleftrightarrow \mathcal{G}(\mathcal{C}_i)$ is a C-M $\mathcal{G}$-module, $\mathrm{a}(\mathcal{G}(\mathcal{C}_i)) < 0$ for $1 \leq \forall i \leq \ell$

$\Longleftrightarrow \mathcal{G}(\mathcal{M})$ is a sequentially C-M $\mathcal{G}$-module, $\{\mathcal{G}(\mathcal{D}_i)\}_{0 \leq i \leq \ell}$ is the dimension filtration of $\mathcal{G}(\mathcal{M})$ and $\mathrm{a}(\mathcal{G}(\mathcal{C}_i)) < 0$ for $1 \leq \forall i \leq \ell$.

□

Seq C-M property in $E^\natural$

Let $R = \sum_{n \geq 0} R_n$ be a $\mathbb{Z}$-graded ring. We put

$$F_n = \sum_{k \geq n} R_k \quad \text{for} \quad \forall n \in \mathbb{Z}.$$

Then F_n is a graded ideal of R, $\mathcal{F} = \{F_n\}_{n \in \mathbb{Z}}$ is a filtration of ideals of R and $F_1 := R_+ \neq R$.

Let E be a graded R-module with $E_n = (0)$ for $\forall n < 0$. Put

$$E_{(n)} = \sum_{k \geq n} E_k \quad \text{for} \quad \forall n \in \mathbb{Z}.$$

Then $E_{(n)}$ is a graded R-submodule of E, $\mathcal{E} = \{E_{(n)}\}_{n \in \mathbb{Z}}$ is an $\mathcal{F}$-filtration of R-submodules of E.

Then we have

$$\underline{\underline{R = \mathcal{G}(\mathcal{F})}} \quad \text{and} \quad \underline{\underline{E = \mathcal{G}(\mathcal{E})}}.$$

Assumption 5.1

- $R = \sum_{n \geq 0} R_n$ a Noetherian $\mathbb{Z}$-graded ring
- $E \neq (0)$ a finitely generated graded R-module with $d = \dim_R E < \infty$

We set

$$\underline{R^\natural := \mathcal{R}(\mathcal{F})} \quad \text{and} \quad \underline{E^\natural := \mathcal{R}(\mathcal{E})}.$$

Lemma 5.2

Then the following assertions hold true.

(1) $R^\natural$ *is a Noetherian ring.*

(2) $E^\natural$ *is a finitely generated graded* $R^\natural$*-module.*

(3) $\dim_{\mathcal{R}'} \mathcal{R}'(\mathcal{E}) = \dim_R E + 1$.

(4) *Suppose that* $\exists P \in \mathrm{Ass}_R E$ *s.t.* $\dim R/P = d$, $F_1 \not\subseteq P$. *Then* $\dim_{R^\natural} E^\natural = \dim_R E + 1$.

Let

$$D_0 = (0) \subsetneq D_1 \subsetneq \ldots \subsetneq D_\ell = E$$

be the dimension filtration of E. We set $C_i = D_i/D_{i-1}$, $d_i = \dim_R D_i$ for $1 \leq \forall i \leq \ell$.

Then D_i is a graded R-submodule of E for $0 \leq \forall i \leq \ell$.

Let $1 \leq i \leq \ell$. Then we get the exact sequence

$$0 \to [D_{i-1}]_{(n)} \to [D_i]_{(n)} \to [C_i]_{(n)} \to 0$$

of graded R-modules for $\forall n \in \mathbb{Z}$.

Therefore

$$0 \to \mathcal{R}(\mathcal{D}_{i-1}) \to \mathcal{R}(\mathcal{D}_i) \to \mathcal{R}(\mathcal{C}_i) \to 0$$

$$0 \to \mathcal{R}'(\mathcal{D}_{i-1}) \to \mathcal{R}'(\mathcal{D}_i) \to \mathcal{R}'(\mathcal{C}_i) \to 0 \quad \text{and}$$

$$0 \to \mathcal{G}(\mathcal{D}_{i-1}) \to \mathcal{G}(\mathcal{D}_i) \to \mathcal{G}(\mathcal{C}_i) \to 0$$

of graded modules, where $\mathcal{D}_i = \{[D_i]_{(n)}\}_{n\in\mathbb{Z}}$, $\mathcal{C}_i = \{[C_i]_{(n)}\}_{n\in\mathbb{Z}}$.

Lemma 5.3

$\{\mathcal{R}'(\mathcal{D}_i)\}_{0\le i\le \ell}$ is the dimension filtration of $\mathcal{R}'(\mathcal{E})$. If $F_1 \not\subseteq \mathfrak{p}$ for $\forall \mathfrak{p} \in \operatorname{Ass}_R E$, then $\{\mathcal{R}(\mathcal{D}_i)\}_{0\le i\le \ell}$ is the dimension filtration of $\mathcal{R}(\mathcal{E})$.

Proposition 5.4

TFAE.

(1) $\mathcal{R}'(\mathcal{E})$ *is a sequentially C-M* $\mathcal{R}'$*-module.*

(2) E *is a sequentially C-M* R*-module.*

Lemma 5.5

Suppose R_0 *is a local ring,* E *is a C-M* R*-module,* $\exists \mathfrak{p} \in \operatorname{Ass}_R E$ *s.t.* $\dim R/\mathfrak{p} = d$, $\mathfrak{p} \not\supseteq F_1$. *Then* $E^\natural$ *is a C-M* $R^\natural$*-module if and only if* $\mathrm{a}(E) < 0$.

Proof (sketch).

Let $P = \mathfrak{m}R + R_+$, where $\mathfrak{m}$ denotes the maximal ideal of R_0. Then $P \supseteq F_1$ and

$$E \cong \mathcal{G}(\mathcal{E}) \cong \mathcal{G}(\mathcal{E}_P), \quad R \cong \mathcal{G} \cong \mathcal{G}(R_P)$$

since $R_+(E_{(n)}/E_{(n+1)}) = (0)$, $R_+(F_n/F_{n+1}) = (0)$ for $\forall n \in \mathbb{Z}$. The assertion comes from the above isomorphisms.

Apply Lemma 5.5, we finally get the following.

Theorem 5.6

Suppose that R_0 is a local ring, E is a sequentially C-M R-module and $\mathfrak{p} \not\supseteq F_1$ for $\forall \mathfrak{p} \in \mathrm{Ass}_R E$. Then TFAE.

(1) *$E^\natural$ is a sequentially C-M $R^\natural$-module.*

(2) $\mathrm{a}(C_i) < 0$ *for* $1 \leq \forall i \leq \ell$.

Application –Stanley-Reisner algebras–

Notation 6.1

- $V = \{1, 2, \dots, n\}$ $(n > 0)$ *a vertex set*
- Δ *a simplicial complex on* V *s.t.* $\Delta \neq \emptyset$
- $\mathcal{F}(\Delta)$ *a set of facets of* Δ
- $m = \sharp\mathcal{F}(\Delta)$ (> 0) *its cardinality*
- $S = k[X_1, X_2, \dots, X_n]$ *a polynomial ring over a field* k
- $I_\Delta = (X_{i_1} X_{i_2} \cdots X_{i_r} \mid \{i_1 < i_2 < \cdots < i_r\} \notin \Delta)$
- $R = k[\Delta] = S/I_\Delta$ *the Stanley-Reisner ring of* Δ

Definition 6.2

A simplicial complex Δ is *shellable*
$\overset{def}{\Longleftrightarrow} m = 1$ or $m > 1$ and $\exists F_1, F_2, \ldots, F_m \in \mathcal{F}(\Delta)$ s.t.
(1) $\mathcal{F}(\Delta) = \{F_1, F_2, \ldots, F_m\}$
(2) $\langle F_1, F_2, \ldots, F_{i-1}\rangle \cap \langle F_i \rangle$ is pure and
$\dim \langle F_1, F_2, \ldots, F_{i-1}\rangle \cap \langle F_i \rangle = \dim F_i - 1$ for $2 \leq \forall i \leq m$.

Theorem 6.3 ([St])

If Δ is shellable, then $R = k[\Delta]$ is a sequentially C-M ring.

Remark 6.4

If Δ is shellable, then we can take a shellable numbering $\mathcal{F}(\Delta) = \{F_1, F_2, \ldots, F_m\}$ s.t. $\dim F_1 \geq \dim F_2 \geq \cdots \geq \dim F_m$.

We now regard $R = \sum_{n \geq 0} R_n$ as a $\mathbb{Z}$-graded ring and put

$$I_n := \sum_{k \geq n} R_k = \mathfrak{m}^n \quad \text{for} \quad \forall n \in \mathbb{Z}$$

where $\mathfrak{m} := R_+ = \sum_{n>0} R_n$. Then $\mathcal{I} = \{I_n\}_{n \in \mathbb{Z}}$ is a $\mathfrak{m}$-adic filtration of R and $I_1 \neq R$.

Proposition 6.5

If Δ is shellable, then $\mathcal{R}'(\mathfrak{m})$ is a sequentially C-M ring.

Remark 6.6

$$\mathfrak{p} \not\supseteq I_1 \text{ for } \forall \mathfrak{p} \in \operatorname{Ass} R \iff F \neq \emptyset \text{ for } \forall F \in \mathcal{F}(\Delta) \iff \Delta \neq \{\emptyset\}.$$

Theorem 6.7

Suppose that Δ is shellable with shellable numbering $\mathcal{F}(\Delta) = \{F_1, F_2, \ldots, F_m\}$ s.t. $\dim F_1 \geq \dim F_2 \geq \cdots \geq \dim F_m$ and $\Delta \neq \{\emptyset\}$. Then TFAE.

(1) $\mathcal{R}(\mathfrak{m})$ *is a sequentially C-M ring.*

(2) $m = 1$ *or* $m \geq 2$*, then* $\dim F_i - 1 > \sharp\mathcal{F}(\Delta_1 \cap \Delta_2)$ *for* $2 \leq \forall i \leq m$*, where* $\Delta_1 = \langle F_1, F_2, \ldots, F_{i-1}\rangle$, $\Delta_2 = \langle F_i \rangle$.

Apply Theorem 6.7, we get the following.

Corollary 6.8

Suppose that $\dim F_m > 2$. *If* $\langle F_1, F_2, \ldots, F_{i-1}\rangle \cap \langle F_i\rangle$ *is a simplex for* $2 \leq \forall i \leq m$, *then* $\mathcal{R}(\mathfrak{m})$ *is a sequentially C-M ring.*

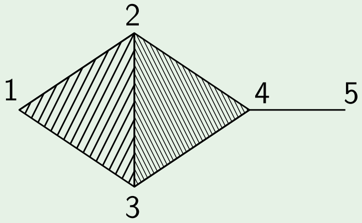
Example 6.9

Let $\Delta = \langle F_1, F_2, F_3 \rangle$, where $F_1 = \{1,2,3\}$, $F_2 = \{2,3,4\}$ and $F_3 = \{4.5\}$. Then Δ is shellable with the numbering $\mathcal{F}(\Delta) = \{F_1, F_2, F_3\}$. Then

$$\langle F_1 \rangle \cap \langle F_2 \rangle\,, \quad \langle F_1, F_2 \rangle \cap \langle F_3 \rangle$$

are simplex, so that $\mathcal{R}(\mathfrak{m})$ is a sequentially C-M ring.

$\Delta =$

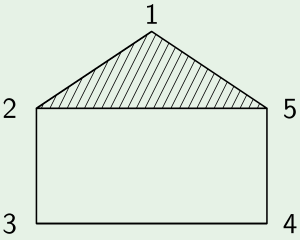


Example 6.10

Let $\Delta = \langle F_1, F_2, F_3, F_4 \rangle$, where $F_1 = \{1, 2, 5\}$, $F_2 = \{2, 3\}$, $F_3 = \{3, 4\}$ and $F_4 = \{4, 5\}$. Then Δ is shellable with the numbering $\mathcal{F}(\Delta) = \{F_1, F_2, F_3, F_4\}$. We put $\Delta_1 = \langle F_1, F_2, F_3 \rangle$, $\Delta_2 = \langle F_4 \rangle$. Then

$$\sharp \mathcal{F}(\Delta_1 \cap \Delta_2) = 2 = \dim F_4 - 1,$$

so that $\mathcal{R}(\mathfrak{m})$ is not a sequentially C-M ring by Theorem 6.7.

$\Delta =$ 

Thank you very much for your attention!

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